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ON THE SHELL EQUATIONS IN COMPLEX FORM*

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INTRODUCTION

This paper is concerned with the problem of formulating the general equations of linear thin shell theory in terms of complex combinations of dependent variables in such a way as to arrive at a fourth order system of partial differential equations. Consideration is limited to elastic, isotropic, homogeneous shells with edge loads only.

Novozhilov was the first to formulate such a system of equations for a shell with an arbitrary middle surface subjected to arbitrarily distributed loads. A full account of his theory is given in [1]. A history of the problem and survey of developments up to 1962 was given by Novozhilov in [2]; therefore, no such survey will be attempted here. Further developments of the Novozhilov theory, principally with respect to the "displacement" form of the equations, boundary conditions, and variational principles are contained in a recently published book by Chernykh [3]. Related results concerned with extensions of the theory to include effects of anisotropy, inhomogeneity, and thermal strains have been obtained by Librescu and by Visarion and Stanescu in [4], [5], and other papers. In the author's understanding, these extensions lead to a system of equations of a mixed type

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which contain the operations of partial differentiation and the taking of complex conjugates. In general the "conjugate terms" are not negligible and cannot be eliminated without raising the "order" of the system above four.

The remarkable symmetry in the complete system of shell equations (including stress-stress function relations) discovered by Goldenveizer and called the static-geometric analogy leads to an almost obvious complex formulation of the equations in the case of vanishing Poisson's ratio ($\nu = 0$). In the case $\nu \neq 0$, difficulties arise. Novozhilov, in effect, derived approximate equilibrium and compatibility equations for which the difficulty is avoided. The validity of the approximate equations is not easy to prove in general, although considerable thought and effort has been devoted to establishing their correctness. As far as this author is aware, the validity of the equations has not yet been definitely disproved in any particular case. Chernykh derived the same system of equations by a different method. He first arrived at a "mixed" system (of the type noted previously) and then obtained equations (through several transformations) in which the conjugate terms appear multiplied by a small parameter; these terms were dropped at each stage of the transformations. However, it is extremely difficult to justify the dropping of apparently small terms in a system of partial differential equations.

In the present paper the equations of shell theory are obtained in complex form (free of conjugate terms) by a method which avoids the necessity of approximating the equilibrium and compatibility equations or any of the equations except the constitutive relations. This is done at the expense of introducing an auxiliary set of partial differential equations for certain "error terms" in the constitutive relations. The order of magnitude of the error terms does not exceed the order of magnitude of the errors inherent in the constitutive relations due to the fundamental hypotheses of thin shell theory. Solution of this set of shell equations divides into two problems: I. the solution of a determinate fourth order system of equations similar to the Novozhilov equations, but with a few more terms, and II, the solution of a set of equations identical in form to the non-homogeneous equilibrium equations for the error terms previously mentioned.

For general cylinders and for shells of revolution, Problem I is reduced (in two ways) to the solution of a single fourth order partial differential equation in a scalar unknown. An alternative form of Problem II is derived and a complete solution of Problem II without quadratures is obtained for spheres and general cylinders.

FUNDAMENTAL EQUATIONS IN DIMENSIONLESS FORM

Let L be a reference length, the length over which significant changes in the dependent variables occur, sometimes called a wave length. Let R be a reference radius of curvature, σ a reference stress, and h the (constant) shell thickness. Let the equation of the middle surface of the shell be

$$\bar{x}^i = \bar{x}^i(\bar{\xi}^\alpha)$$

where $\bar{x}^i$ and $\bar{\xi}^\alpha$ have dimensions of length, and a change in $\bar{\xi}^\alpha$ of $O(R)$ corresponds to a distance of $O(R)$. Introduce the following dimensionless variables:

$$\bar{x}^i = Rx^i \qquad \bar{g}_{\alpha\beta} = g_{\alpha\beta} \quad \text{(first fundamental form)}$$

$$\bar{\xi}^\alpha = L\xi^\alpha \qquad R\bar{b}_{\alpha\beta} = b_{\alpha\beta} \quad \text{(second " ")}$$

A few of the familiar equations of surfaces in dimensionless form are as follows

$$ds^2 = g_{\alpha\beta} d\xi^\alpha \, d\xi^\beta \qquad x^i_{,\alpha} \, x^i_{,\beta} = \beta\mu \, g_{\alpha\beta}$$
$$n^i_{,\alpha} = b^\gamma_\alpha \, x^i_{,\gamma} \qquad x^i_{,\alpha\beta} = -\beta\mu \, b_{\alpha\beta} \, n^i$$

where n^i is the unit normal to the surface, $b_{\alpha\beta}$, the second fundamental form, differs in sign from the usual definition, and a comma denotes covariant differentiation. The dimensionless parameters β and μ are defined as follows

$$\mu = \frac{L^2}{Rh} \sqrt{12(1 - \nu^2)} \qquad \beta = \frac{h}{R\sqrt{12(1 - \nu^2)}} \quad .$$

Dimensionless stress and strain measures, displacements and stress functions (without a tild~~a~~e) are related to the dimensional variables as follows

$$\tilde{N}_{\alpha\beta} = \sigma h N_{\alpha\beta} \qquad \tilde{E}_{\alpha\beta} = \frac{\sigma}{E} E_{\alpha\beta}$$
$$\tilde{M}_{\alpha\beta} = \frac{\sigma h L^2}{\mu R} M_{\alpha\beta} \qquad \tilde{K}_{\alpha\beta} = \frac{\sigma R \mu}{E L^2} K_{\alpha\beta}$$
$$\tilde{u}_\alpha = \frac{\sigma L}{E} u_\alpha \qquad \tilde{\chi}_\alpha = \frac{\sigma h L^2}{\mu R} \chi_\alpha$$
$$\tilde{w} = \frac{\mu \sigma R}{E} w \qquad \tilde{\psi} = \sigma h L^2 \psi \quad .$$

Two special notations which permit the shell equations to be written in a more compact form are introduced next. Define a tensor $B_{\alpha\beta}$ in terms of $b_{\alpha\beta}$ as follows

$$B_{\alpha\beta} = \frac{1}{2}(\epsilon_{\alpha\gamma} \, b^\gamma_\beta + \epsilon_{\beta\gamma} \, b^\gamma_\alpha)$$

where $\epsilon_{\alpha\beta}$ is the usual permutation tensor. For any second order tensor define a "bar" operation as follows

$$\overline{T}_{\alpha\beta} = \varepsilon_{\alpha\lambda}\varepsilon_{\beta\mu} \; T^{\lambda\mu} \equiv g_{\alpha\beta} \, T^{\lambda}_{\lambda} - T_{\beta\alpha}$$

Note that $\overline{\overline{T}}_{\alpha\beta} = T_{\alpha\beta}$, $\overline{T}^{\alpha}_{\alpha} = T^{\alpha}_{\alpha}$, $\overline{g}_{\alpha\beta} = g_{\alpha\beta}$, $\overline{B}_{\alpha\beta} = - B_{\alpha\beta}$. Hereafter a bar over a second order tensor will always denote this operation.

The following field equations of linear thin shell theory are taken from [6], [7], [8] and put in dimensionless form.

Strain-displacement relations

$$\begin{aligned} E_{\alpha\beta} &= \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + \mu b_{\alpha\beta} \, w \\ K_{\alpha\beta} &= \frac{1}{2}(\phi_{\alpha,\beta} + \phi_{\beta,\alpha}) + \beta \, B_{\alpha\beta} \, \omega \\ \phi_{\alpha} &= - w_{,\alpha} + \beta \, b^{\gamma}_{\alpha} \, u_{\gamma} \\ \omega \; \cancel{\psi} &= \frac{1}{2} \, \varepsilon^{\alpha\beta} \, u_{\beta,\alpha} \end{aligned} \tag{1}$$

Stress-stress function relations

$$\begin{aligned} \overline{M}_{\alpha\beta} &= \frac{1}{2}(\chi_{\alpha,\beta} + \chi_{\beta,\alpha}) + \mu \, b_{\alpha\beta}\psi \\ \overline{N}_{\alpha\beta} &= - \frac{1}{2}(\theta_{\alpha,\beta} + \theta_{\beta,\alpha}) - \beta \, B_{\alpha\beta}\Omega \\ \theta_{\alpha} &= - \psi_{,\alpha} + \beta \, b^{\gamma}_{\alpha} \, \chi_{\gamma} \\ \Omega &= \frac{1}{2} \, \varepsilon^{\alpha\beta}\chi_{\beta,\alpha} \end{aligned} \tag{2}$$

Equilibrium equations

$$\begin{aligned} N^{\alpha\beta}{}_{,\beta} + \beta \, b^{\alpha}_{\gamma} \, M^{\gamma\beta}{}_{,\beta} - \frac{1}{2} \, \beta \, \varepsilon^{\alpha\beta}(B_{\gamma\delta} \, M^{\gamma\delta})_{,\beta} &= 0 \\ M^{\alpha\beta}{}_{,\alpha\beta} - \mu \, b_{\alpha\beta} \, N^{\alpha\beta} &= 0 \end{aligned} \tag{3}$$

Compatibility equations

$$\begin{aligned} \overline{K}^{\alpha\beta}{}_{,\beta} - \beta \, b^{\alpha}_{\gamma} \, \overline{E}^{\gamma\beta}{}_{,\beta} + \frac{1}{2} \, \beta \, \varepsilon^{\alpha\beta}(B_{\gamma\delta}\overline{E}^{\gamma\delta})_{,\beta} &= 0 \\ \overline{E}^{\alpha\beta}{}_{,\alpha\beta} + \mu \, b_{\alpha\beta} \, \overline{K}^{\alpha\beta} &= 0 \end{aligned} \tag{4}$$

Constitutive relations

$$
\begin{aligned}
\tilde{E}_{\alpha\beta} &= -(1+\nu)N_{\alpha\beta} + g_{\alpha\beta}\,N^{\gamma}_{\gamma} \\
M_{\alpha\beta} &= -(1-\nu)K_{\alpha\beta} + g_{\alpha\beta}\,\tilde{K}^{\gamma}_{\gamma}
\end{aligned} \tag{5}
$$

In the shell theory set down here the tensor measures of stress and strain are symmetric. The similarity in form between pairs of equations is, of course, a manifestation of the static-geometric analogy.

The reference length L appearing in the parameter μ and elsewhere is meant to be chosen in such a way that differentiation does not change the order of magnitude of any of the fundamental variables $N^{\alpha\beta}$, w etc. In some cases (notably for certain solutions of the equations of cylindrical shells) the variables change more rapidly in one direction than in another. In such a case L should be the shorter of the two "wavelengths" so that differentiation will not increase the order of magnitude of a variable. It seems unlikely that the variables $N^{\alpha\beta}$, w etc., can change much less rapidly than the surface variables, $b_{\alpha\beta}$, $B_{\alpha\beta}$ etc.; in any case it will generally be assumed that differentiation of these surface variables does not increase their order of magnitude. The reference stress σ can be chosen such that at least one of the components of $N^{\alpha\beta}$ or $M^{\alpha\beta}$ is about unity (over some region of the middle surface of dimension L, usually) and none of the components are much larger than unity.

COMPLEX COMBINATIONS

Let $N'^{\alpha\beta}$, u'_{α} etc., be some solution to the shell equations and let $N''^{\alpha\beta}$, u''_{α} etc., be some other unrelated solution. Let

$$
N^{\alpha\beta} = N'^{\alpha\beta} + i\,N''^{\alpha\beta} \tag{6}
$$

and similarly for all other variables. The Eqs.(1) to (5), now in terms of complex-valued variables, hold as they stand. Of course, nothing has been gained by this operation. However, if some relation can be introduced between the primed and double-primed solutions, then there is the possibility of some gain. To see how this might be so, consider the case of Poisson's ratio ν equal to zero, and introduce the additional equation

$$N^{\alpha\beta} = i\,\overline{K}^{\alpha\beta} \quad . \tag{7}$$

From (5) and (7) follows

$$M^{\alpha\beta} = -\,i\,\overline{E}^{\alpha\beta} \tag{8}$$

and one can obviously set $w = i\psi$ and $u_\alpha = i\,\chi_\alpha$. The number of field equations (1 to 5) is cut in half because the equations become identical by pairs, and the order of the system is reduced from eight to four. By separating real and imaginary parts in Eq.(7), one finds

$$N''_{\alpha\beta} = \overline{K}'_{\alpha\beta} \quad ; \quad \overline{K}''_{\alpha\beta} = -\,N'_{\alpha\beta}$$

so

$$N_{\alpha\beta} = N'_{\alpha\beta} + i\,\overline{K}'_{\alpha\beta}$$

$$\overline{K}_{\alpha\beta} = \overline{K}'_{\alpha\beta} - i\,N'_{\alpha\beta}$$

and similarly for other quantities. Thus, the introduction of the relation (7) is equivalent to the introduction of dependent variables which are complex combinations of quantities which are static-geometric analogs of each other. The shell equations were reduced in this way by Novozhilov in [1]. The fact that the additional relation (7) does not result in an insoluble overdetermination of the system of shell equations is obviously due to the static-geometric analogy (which suggests just such a complex combination of the variables). An immediate difficulty is that $\nu = 0$ is not generally a useful approximation.

Suppose that (7) is retained in the case $\nu \neq 0$, then from (5) there follows (instead of (8))

$$M_{\alpha\beta} = -\,i\,\overline{E}_{\alpha\beta} + 2\nu\,\overline{K}_{\alpha\beta} \quad . \tag{9}$$

One can easily verify that the system of Eqs.(1) to (5) plus the relations (7) and (9) is inconsistent. However, consider the Vlosov approximate equations which are obtained from (1) to (5) by setting $\beta = 0$ (to which should be appended the rule $\epsilon^{\alpha\beta} T_{\gamma,\alpha\beta} = 0$). By (7) the equation $N^{\alpha\beta}{}_{,\beta} = 0$ becomes $i \overline{K}^{\alpha\beta}{}_{,\beta} = 0$ which is true. The equation

$$M^{\alpha\beta}{}_{,\alpha\beta} - \mu b_{\alpha\beta} N^{\alpha\beta} = 0$$

becomes, by (7) and (9)

$$i \overline{E}^{\alpha\beta}{}_{,\alpha\beta} + 2\nu \overline{K}^{\alpha\beta}{}_{,\alpha\beta} - i\mu\, b_{\alpha\beta} \overline{K}^{\alpha\beta} = 0$$

which is the same as

$$\overline{E}^{\alpha\beta}{}_{,\alpha\beta} + \mu b_{\alpha\beta} \overline{K}^{\alpha\beta} = 0$$

since $\overline{K}^{\alpha\beta}{}_{,\alpha\beta} = 0$.

Thus the compatibility equations are equivalent to the equilibrium equations. The relations $\psi = - i w$ and $\chi_\alpha = - iu_\alpha - 2\nu\ w_{,\alpha}$ between complex stress functions and complex displacements also holds. The complete system of Vlosov equations can thus be put in complex form with no further approximations and obviously half of the equations and variables could be discarded to reduce the order of the system by half. The reduction to a single fourth order equation in w or ψ is well known.

In the case of a spherical shell ($b_{\alpha\beta} = g_{\alpha\beta}$, $B_{\alpha\beta} = 0$) the introduction of a slight modification to the relation (7) leads to a similar reduction of the exact system of equations. Put

$$N_{\alpha\beta} = i \lambda \overline{K}_{\alpha\beta} \tag{10}$$

where

$$\lambda^2 - 1 - 2i\, \beta\, \nu\, \lambda = 0 \quad .$$

Since β is small there is a root $\lambda \approx 1$. The relation (9) is replaced by

$$\overline{E}_{\alpha\beta} = i \lambda M_{\alpha\beta} - 2i \lambda \nu \overline{K}_{\alpha\beta} \tag{11}$$

For λ a root of the above quadratic equation, one can easily verify that the equilibrium and compatibility equations are equivalent to each other, and moreover the following relations hold

$$w = i \lambda \psi \quad , \quad \chi_\alpha = - i \lambda u_\alpha - 2\nu w_{;\alpha}$$

Without much difficulty the system of equations can be reduced to a single fourth order equation similar to one derived by Koiter [9]. Unfortunately, in the case of a general shell; (10) does not lead to a similar result for any value of λ .

Novozhilov [1] in effect derives approximate sets of equilibrium and compatibility equations for which the introduction of (7) does not lead to contradictions. Novozhilov, of course, bases his derivations on his set of shell equations which differ (but in no essential way) from (1) to (4). In his derivation of approximate equations for the complex theory, the equation of moment equilibrium about the normal to the middle surface is not enforced. Such an approximation is known to lead to difficulties with respect to accuracy in some cases, notably in the case of helicoidal shells [10],[11],[12],[13]. In any case, some approximations in the equilibrium and compatibility equations are necessary if one wishes to put the shell equations in complex form and retain relation (7). However, in [13] Koiter has shown that the exact equilibrium and compatibility equations cannot, in general, be simplified. That is to say, no term in them is always negligible in all possible cases. Since the Novozhilov equations have been tested in practice and subjected to careful study, there seems to be little doubt that they are sufficiently accurate for most practical cases, at least for general cylinders and shells of revolution. There does, however, seem to be room for reasonable doubt that they are accurate (to within the usual limits of shell theory) in all cases, and in particular for certain problems of the helicoidal shell.

In view of the foregoing it is evident that the relation (7) must be nearly the correct one to introduce if one wishes to put the shell equations in complex form without introducing essential errors, if indeed this is possible. In the present paper a modification of (7) is introduced which leads to the desired result without approximation of the equilibrium or compatibility equations.

FORMULATION OF COMPLEX EQUATIONS IN THE GENERAL CASE

Since $N^{\alpha\beta} - i\,\overline{K}^{\alpha\beta}$ is expected to be small (and for other reasons) it is convenient to introduce the "double" complex combinations of variables given below

$$P^{\alpha\beta} = N^{\alpha\beta} + i\,\overline{K}^{\alpha\beta} \qquad \overset{*}{P}{}^{\alpha\beta} = N^{\alpha\beta} - i\,\overline{K}^{\alpha\beta}$$
$$Q^{\alpha\beta} = M^{\alpha\beta} - i\,\overline{E}^{\alpha\beta} \qquad \overset{*}{Q}{}^{\alpha\beta} = M^{\alpha\beta} + i\overline{E}^{\alpha\beta} \quad . \tag{12}$$

(Because of the way in which the stress and strain measures were made dimensionless $P^{\alpha\beta}$ and $Q^{\alpha\beta}$ must be $O(1)$. $P^{\alpha\beta} = O(1)$ means that at least one of the components of $P^{\alpha\beta}$ has a real or imaginary part which is about unity and no component has a real or imaginary part much greater than unity. In general the real and imaginary parts of $P^{\alpha\beta}$ can differ in order of magnitude. In equation (13) below an apparently small term cannot be dropped unless this term is negligible in both the real and imaginary parts of the equation. In the Vlasov case the real and imaginary parts of $P^{\alpha\beta}$ and $Q^{\alpha\beta}$ are $O(1)$ and the β terms can be dropped.)

The (complex) equilibrium and compatibility equations (3) and (4) combine together to give

$$P^{\alpha\beta}{}_{,\beta} + \beta\; b^{\alpha}_{\gamma}\, Q^{\gamma\beta}{}_{,\beta} - \tfrac{1}{2}\beta\; \varepsilon^{\alpha\beta}(B_{\gamma\delta}Q^{\gamma\delta})_{,\beta} = 0 \tag{13}$$
$$Q^{\alpha\beta}{}_{,\alpha\beta} - \mu\, b_{\alpha\beta}\, P^{\alpha\beta} = 0$$

and the same pair of equations in terms of $\overset{*}{P}{}^{\alpha\beta}$ and $\overset{*}{Q}{}^{\alpha\beta}$, hereafter referred to as (13)* . Also define

$$W = \psi - i\,w \qquad X_\alpha = \chi_\alpha - i\,u_\alpha$$
$$\Phi_\alpha = \theta_\alpha - i\,\phi_\alpha \qquad z = \Omega - i\,\omega \tag{14}$$

along with $W^* = \psi + i\,w$, etc.

The (complex) stress-stress function relations (2) and strain displacement relations (1) combine in the following form

$$\overline{P}_{\alpha\beta} = -\frac{1}{2}(\Phi_{\alpha,\beta} + \Phi_{\beta,\alpha}) - \beta\, B_{\alpha\beta} z$$
$$\overline{Q}_{\alpha\beta} = \frac{1}{2}(X_{\alpha,\beta} + X_{\beta,\alpha}) + \mu\, b_{\alpha\beta} W \tag{15}$$

where

$$\Phi_\alpha = -W_{,\alpha} + \beta\, b^\gamma_\alpha X_\gamma$$

and

$$z = \frac{1}{2}\varepsilon^{\alpha\beta} X_{\beta,\alpha}$$

and the same equations with stars.

In place of the constitutive relations (5) introduce the following modified relations

$$\overset{*}{P}{}^{\alpha\beta} = i\,\beta\,\nu\, R^{\alpha\beta} \tag{16}$$
$$\overset{*}{Q}{}^{\alpha\beta} = -i\,\nu\, P^{\alpha\beta} + i\,\beta\,\nu\, S^{\alpha\beta} \tag{17}$$
$$Q^{\alpha\beta} = -i\,\overline{P}{}^{\alpha\beta} \tag{18}$$

in which $R^{\alpha\beta}$ and $S^{\alpha\beta}$ are, for the present, arbitrary symmetric tensors except for the restriction $R^{\alpha\beta} = O(1)$ or $O(1/\mu)$, whichever is greater, and the same for $S^{\alpha\beta}$. If, in (16) to (18) one sets $\beta = 0$ and solves for $\overline{E}_{\alpha\beta}$ in terms of $N_{\alpha\beta}$ and $M_{\alpha\beta}$ in terms of $-\overline{K}_{\alpha\beta}$ the results are exactly the constitutive relations (5). Equation (16) becomes equation (7). With $\beta \neq 0$ the constitutive relations implied by (16) to (18) are (5) with error terms of $O(\beta, \beta/\mu)$. Novozhilov [1] and Koiter [6] have shown that it is permissible to introduce absolute errors of this order in the (dimensionless) constitutive relations without impairing the accuracy of the system of shell equations because the constitutive relations already contain errors of this order from the fundamental assumptions. With $\beta \neq 0$ equation (16) is a modification of the relation (7). As will be shown, the tensors $R_{\alpha\beta}$ and $S_{\alpha\beta}$ can be chosen such that the system of shell equations is not overdetermined in an inconsistent manner by the introduction of one extra tensor relation. If either $\nu = 0$ or $\beta = 0$ the tensors $R_{\alpha\beta}$ and $S_{\alpha\beta}$ drop out of the relations (16) to (18) and the corresponding system of shell equations is consistent, as has already been shown.

Any set of four tensors $P^{\alpha\beta}$, $Q^{\alpha\beta}$, $\overset{*}{P}{}^{\alpha\beta}$ and $\overset{*}{Q}{}^{\alpha\beta}$ which satisfy (13) and (13)* are equivalent to a solution of the shell equations provided $R^{\alpha\beta}$ and $S^{\alpha\beta}$ can be found such that (16) to (18) are satisfied with restrictions on the order of magnitude of $R^{\alpha\beta}$ and $S^{\alpha\beta}$ observed. The problem splits into two parts: problem I, the determination of $P^{\alpha\beta}$; problem II, the determination of $R^{\alpha\beta}$ and $S^{\alpha\beta}$.

<u>Problem I.</u> Use equation (18) to eliminate the variable $Q^{\alpha\beta}$ from equations (13). The result is

$$\begin{aligned} & P^{\alpha\beta}{}_{,\beta} + i\beta\, b^{\alpha}_{\gamma}\, P^{\gamma\beta}{}_{,\beta} - i\beta\, b^{\alpha\beta}\, P_{,\beta} - \frac{1}{2}\, i\beta\varepsilon^{\alpha\beta}(B_{\gamma\delta}P^{\gamma\delta})_{,\beta} = 0 \\ & i\, P^{\alpha\beta}{}_{,\alpha\beta} - i\, \nabla^2 P - \mu b_{\alpha\beta}\, P^{\alpha\beta} = 0 \end{aligned} \tag{19}$$

in which $P = P^{\alpha}_{\alpha}$ and $\nabla^2 P = g^{\alpha\beta}\, P_{,\alpha\beta}$. To obtain these equations the following relations have been used.

$$\overline{P}^{\alpha\beta} = g^{\alpha\beta}\, P - P^{\alpha\beta}\ ,\ B_{\alpha\beta}\, \overline{P}^{\alpha\beta} = \overline{B}_{\alpha\beta}\, P^{\alpha\beta} = -\, B_{\alpha\beta}\, P^{\alpha\beta}\ .$$

By obvious manipulations (19) reduces to the following (with β^2 terms dropped)

$$\begin{aligned} & P^{\alpha\beta}{}_{,\beta} - i\,\beta\, b^{\alpha\beta}\, P_{,\beta} - \frac{1}{2}\, i\beta\varepsilon^{\alpha\beta}(B_{\gamma\delta}\, P^{\gamma\delta})_{,\beta} = 0 \\ & \nabla^2 P - i\,\mu\, b_{\alpha\beta}\, P^{\alpha\beta} - i\beta(b^{\alpha\beta}P_{,\beta})_{,\alpha} = 0\ . \end{aligned} \tag{20}$$

These equations constitute a determinate fourth order system of equations for the components of the tensor $P^{\alpha\beta}$. If the last term in each of (20) is omitted, then these equations are identical in form (but for a different variable) to the complex shell equations derived by Novozhilov. In certain cases (to be discussed later) these equations have been reduced to a single fourth order equation in a scalar variable.

Problem II. Replace $\overset{*}{P}{}^{\alpha\beta}$ and $\overset{*}{Q}{}^{\alpha\beta}$ in equations (13)* by their expressions in terms of $R^{\alpha\beta}$, $S^{\alpha\beta}$, and $P^{\alpha\beta}$ from (16) and (17). The result is

$$\begin{gathered} R^{\alpha\beta}{}_{,\beta} + \beta\, b^\alpha_\gamma S^{\gamma\beta}{}_{,\beta} - \frac{1}{2}\beta\varepsilon^{\alpha\beta}(B_{\gamma\delta}S^{\gamma\delta})_{,\beta} - b^\alpha_\gamma P^{\gamma\beta}{}_{,\beta} + \frac{1}{2}\varepsilon^{\alpha\beta}(B_{\gamma\delta}P^{\gamma\delta})_{,\beta} = 0 \\ \beta(S^{\alpha\beta}{}_{,\alpha\beta} - \mu\, b_{\alpha\beta} R^{\alpha\beta}) - P^{\alpha\beta}{}_{,\alpha\beta} = 0 \quad . \end{gathered} \tag{21}$$

These equations can be reduced somewhat by the use of the first of equations (20) with the result

$$\begin{gathered} R^{\alpha\beta}{}_{,\beta} + \beta\, b^\alpha_\gamma S^{\gamma\beta}{}_{,\beta} - \frac{1}{2}\beta\, \varepsilon^{\alpha\beta}(B_{\gamma\delta}S^{\gamma\delta})_{,\beta} = -\frac{1}{2}\varepsilon^{\alpha\beta}(B_{\gamma\delta}P^{\gamma\delta})_{,\beta} \\ S^{\alpha\beta}{}_{,\alpha\beta} - \mu\, b_{\alpha\beta} R^{\alpha\beta} = i(b^{\alpha\beta}P_{,\beta})_{,\alpha} \quad . \end{gathered} \tag{22}$$

These equations have been simplified by the omission of certain β and β^2 terms on the right-hand side, but without claiming that these terms are always negligible. The solution of equations (22) for $R^{\alpha\beta}$ and $S^{\alpha\beta}$ is equivalent to the determination of a particular solution to the non-homogeneous equilibrium equations (regarding $P^{\alpha\beta}$ as known from the solution of (20)). Since the original equations were made dimensionless in such a way that $P^{\alpha\beta}$ is $O(1)$, there should be a particular solution to the equations (22) with $R^{\alpha\beta}$ and $S^{\alpha\beta}$ $O(1,1/\mu)$.

In a later section Problem II is recast in another form and exact solutions without quadratures are obtained in the cases of a sphere and a general cylinder.

FURTHER RESULTS ON PROBLEM I

The equations (20) for $P^{\alpha\beta}$ probably cannot be reduced to a single fourth order equation in a scalar variable in the general case. In [1] Novozhilov has reduced the equations for a general cylinder to a single equation in a variable roughly corresponding to the variable P and the equations for a shell of revolution to two coupled second-order equations. Different reductions to a fourth order equation have been made by Koiter [9] for a sphere and by Simmonds [14] for a circular cylinder. In the present section equations (20) are reduced to a fourth order equation (in two ways) for the cases of a general cylinder and a shell of revolution (except for a sphere). The sphere is treated as a special case. Before proceeding certain geometrical results are developed.

Geometrical Note

Except in the case of a sphere, any second order tensor on a surface can be represented in the following form

$$T^{\alpha\beta} = \lambda^1 g^{\alpha\beta} + \lambda^2 b^{\alpha\beta} + \lambda^3 B^{\alpha\beta} + \lambda^4 \varepsilon^{\alpha\beta} \tag{23}$$

where the λ^i are scalars. The elimination process to be performed is based on the representation of $P^{\alpha\beta}$ in a form similar to this. A modification of (23), possible for a certain class of shells, leads to somewhat simpler results, but is in no way essential to the elimination process. The modified form of (23) for $P^{\alpha\beta}$ is as follows

$$P^{\alpha\beta} = \lambda^1 g^{\alpha\beta} + \lambda^2 \dot{b}^{\alpha\beta} + \lambda^3 \dot{B}^{\alpha\beta} \tag{24}$$

in which $\dot{B}^{\alpha\beta} = \tau B^{\alpha\beta}$ and $\dot{b}^{\alpha\beta} = \tau (b^{\alpha\beta} - \rho g^{\alpha\beta})$ where $\rho = \frac{1}{2} b^\alpha_\alpha$ is the mean curvature and where τ is determined in such a way that $\dot{b}^{\alpha\beta}{}_{,\beta} = 0$ and $\dot{B}^{\alpha\beta}{}_{,\beta} = 0$. Since $B^{\alpha\beta} = \varepsilon^{\alpha\gamma}(b^\beta_\gamma - \rho\delta^\beta_\gamma)$ the problem is the same for either tensor. Consider

$$[\tau(b^{\alpha\beta} - \rho g^{\alpha\beta})]_{,\beta} = \tau_{,\beta}(b^{\alpha\beta} - \rho g^{\alpha\beta}) + \tau\, g^{\alpha\beta}\, \rho_{,\beta} = 0 \quad . \tag{25}$$

Use $(b^{\gamma}_{\alpha} - \rho\delta^{\gamma}_{\alpha})(b^{\alpha\beta} - \rho g^{\alpha\beta}) = (\rho^2 - \kappa)g^{\beta\gamma}$ where κ is the Gaussian curvature to obtain

$$\tau_{,\alpha} + \frac{1}{\rho^2 - \kappa}(b^{\gamma}_{\alpha} - \rho\delta^{\gamma}_{\alpha})\rho_{,\gamma}\,\tau = 0 \; . \tag{26}$$

The condition of integrability of this equation for τ is

$$\varepsilon^{\alpha\beta}\left[\frac{1}{\rho^2 - \kappa}(b^{\gamma}_{\alpha} - \rho\delta^{\gamma}_{\alpha})\rho_{,\gamma}\right]_{,\beta} = 0$$

or

$$\varepsilon^{\alpha\beta}(b^{\gamma}_{\alpha} - \rho\delta^{\gamma}_{\alpha})\left(\frac{\rho_{,\gamma}}{\rho^2 - \kappa}\right)_{,\beta} \equiv -B^{\beta\gamma}\left(\frac{\rho_{,\gamma}}{\rho^2 - \kappa}\right)_{,\beta} = 0 \quad . \tag{27}$$

This condition is satisfied for general cylinders and cones, surfaces of revolution, and any surface of constant mean curvature. For these cases

$$(\log \tau)_{,\alpha} = -\frac{1}{\rho^2 - \kappa}(b^{\gamma}_{\alpha} - \rho\delta^{\gamma}_{\alpha})\rho_{,\gamma} \; . \tag{28}$$

For general cylinders and shells of revolution in lines-of-curvature coordinates τ , ρ , κ etc., are functions of one coordinate only and (28) is easily integrated. For cylinders $\tau = 1/\rho$, for shells of revolution $\tau = 1/r^2\sqrt{\rho^2 - \kappa}$ where r is the radius of a parallel circle ($R_2 \sin\theta$ in the notation of [1]). A few useful formulas involving $\dot{b}_{\alpha\beta}$ etc., are as follows

$$\begin{aligned} \zeta\, \dot{b}_{\alpha\gamma}\, \dot{b}^{\gamma}_{\beta} &= g_{\alpha\beta} & \qquad \varepsilon_{\alpha\gamma}\, \dot{b}^{\gamma}_{\beta} &= \dot{B}_{\alpha\beta} \\ \zeta\, \dot{B}_{\alpha\gamma}\, \dot{B}^{\gamma}_{\beta} &= g_{\alpha\beta} & \qquad \dot{B}^{\gamma}_{\alpha}\, \varepsilon_{\gamma\beta} &= \dot{b}_{\alpha\beta} \\ \zeta\, \dot{B}_{\alpha\gamma}\, \dot{b}^{\gamma}_{\beta} &= \varepsilon_{\alpha\beta} & \qquad \dot{b}^{\alpha}_{\alpha} &= 0 \end{aligned}$$

where $\qquad \zeta^{-1} = \tau^2(\rho^2 - \kappa) \qquad\qquad \dot{B}^{\alpha}_{\alpha} = 0$:

Characteristic Equation in P . Put (see (19))

$$T^{\alpha} = P^{\alpha\beta}{}_{,\beta} + i\,\beta\, b^{\alpha}_{\gamma}\, P^{\gamma\beta}{}_{,\beta} - i\,\beta\, b^{\alpha\beta}\, P_{,\beta} - \frac{1}{2}\, i\beta\, \varepsilon^{\alpha\beta}(B_{\gamma\delta}P^{\gamma\delta})_{,\beta} \tag{30}$$

$$T = i\, P^{\alpha\beta}{}_{,\alpha\beta} - i\,\nabla^2 P - \mu\, b_{\alpha\beta}\, P^{\alpha\beta} \tag{31}$$

and define

$$\overline{T}^{\alpha} \equiv T^{\alpha} - i\beta\, b^{\alpha}_{\gamma}\, T^{\gamma} = P^{\alpha\beta}{}_{,\beta} - i\,\beta\, b^{\alpha\beta}\, P_{,\beta} - \frac{1}{2}\, i\beta\, \varepsilon^{\alpha\beta}(B_{\gamma\delta}\, P^{\gamma\delta})_{,\beta} \tag{32}$$

in which certain β^2 terms have been omitted. From (31) and (32)

$$T - i\,\overline{T}^{\alpha}{}_{,\alpha} = -\, i\,\nabla^2 P - \mu\, b_{\alpha\beta}\, P^{\alpha\beta} - \beta(b^{\alpha\beta}\, P_{,\beta})_{,\alpha} \quad . \tag{33}$$

Now put

$$P^{\alpha\beta} = \lambda^1\, g^{\alpha\beta} + \lambda^2\, \dot{b}^{\alpha\beta} + \lambda^3\, \dot{B}^{\alpha\beta} \tag{34}$$

in which

$$\begin{aligned} \lambda^1 &= \frac{1}{2}\, P \\ \lambda^2 &= \frac{1}{2}\, \tau\zeta(b_{\alpha\beta}P^{\alpha\beta} - \rho P) \\ \lambda^3 &= \frac{1}{2}\, \tau\zeta\, B_{\alpha\beta}P^{\alpha\beta} \quad . \end{aligned} \tag{35}$$

From (32) and (34)

$$\overline{T}^{\alpha} = g^{\alpha\beta}\, \lambda^1{}_{,\beta} + \dot{b}^{\alpha\beta}\, \lambda^2{}_{,\beta} + \dot{B}^{\alpha\beta}\, \lambda^3{}_{,\beta} - i\beta\, b^{\alpha\beta}\, P_{,\beta} - i\beta\, \varepsilon^{\alpha\beta}[(\tau\zeta)^{-1}\lambda^3]_{,\beta} \quad . \tag{36}$$

Multiply this equation by $\zeta\dot{b}^{\gamma}_{\alpha}$ to obtain

$$\zeta\dot{b}^{\gamma}_{\alpha}\, \overline{T}^{\alpha} = \zeta\, \dot{b}^{\beta\gamma}\, \lambda^1{}_{,\beta} + g^{\beta\gamma}\, \lambda^2{}_{,\beta} + \varepsilon^{\beta\gamma}\lambda^3{}_{,\beta} - i\beta\zeta\, \dot{b}^{\gamma}_{\alpha}\, b^{\alpha\beta}P_{,\beta} + i\beta\zeta\, \dot{B}^{\beta\gamma}[(\tau\zeta)^{-1}\lambda^3]_{,\beta} \tag{37}$$

The first order term in λ^3 can now be eliminated by a differentiation.

$$(\zeta\dot{b}^{\gamma}_{\alpha}\, \overline{T}^{\alpha})_{,\gamma} = \dot{b}^{\beta\gamma}(\zeta\lambda^1{}_{,\beta})_{,\gamma} + \nabla^2\lambda^2 - i\beta(\zeta\dot{b}^{\gamma}_{\alpha}\, b^{\alpha\beta}P_{\beta})_{,\gamma} + i\beta\dot{B}^{\beta\gamma}\{\zeta[(\tau\zeta)^{-1}\lambda^3]_{,\beta}\}_{,\gamma} \tag{38}$$

The last term in (38) is equal to the following

$$i\,\beta\dot{B}^{\beta\gamma}\quad \{\tau\zeta(\frac{1}{\tau^2\zeta}\,\lambda^3{}_{,\beta})_{,\gamma} + \lambda^3\,[\zeta(\frac{1}{\tau\zeta})_{,\beta}\,]_{,\gamma}\} \quad .$$

The term with $\lambda^3{}_{,\beta}$ can be eliminated as follows

$$(\zeta \dot{b}^{\gamma}_{\alpha} \overline{T}^{\alpha})_{,\gamma} - i\beta\tau\zeta(\frac{1}{\tau^2\zeta}\overline{T}^{\alpha})_{,\alpha} = \dot{b}^{\beta\gamma}(\zeta\lambda^1{}_{,\beta})_{,\gamma} + \nabla^2\lambda^2 - i\beta(\zeta\dot{b}^{\gamma}_{\alpha}\, b^{\alpha\beta}P_{,\beta})_{,\gamma}$$
$$- i\beta\tau\zeta g^{\alpha\beta}(\frac{1}{\tau^2\zeta}\lambda^1{}_{,\beta})_{,\alpha} - i\beta\tau\zeta\dot{b}^{\alpha\beta}(\frac{1}{\tau^2\zeta}\lambda^2{}_{,\beta})_{,\alpha} + i\beta\dot{b}^{\beta\gamma}[\zeta(\frac{1}{\tau\zeta})_{,\beta}]_{,\gamma}\lambda^3 . \tag{39}$$

The last term here, the λ^3 term, cannot in general be eliminated without one more differentiation of this equation, and this would lead finally to a characteristic equation in P of order higher than the fourth. However, for general cylinders and shells of revolution, this term vanishes identically and the elimination of λ^3 is complete except for β^2 terms which are certainly negligible. For later reference note here that a variation of (39) is obtained by multiplying (38) by any $O(\beta)$ scalar and adding to (39). This operation does not reintroduce a λ^3 term except to $O(\beta^2)$.

From (33) and (35) it follows that

$$\lambda^2 = -\frac{1}{2}\tau\zeta\rho\, P - \frac{i}{2\mu}\tau\zeta\nabla^2 P - \frac{\beta}{2\mu}\tau\zeta(b^{\alpha\beta}P_{,\beta})_{,\alpha} + \frac{i}{2\mu}\tau\zeta\overline{T}^{\alpha}{}_{,\alpha} - \frac{1}{2\mu}\tau\zeta\, T . \tag{40}$$

Equation (40)is now used to eliminate λ^2 from (39). The result is the following

$$\nabla^2(\tau\zeta\overline{T}^{\alpha}{}_{,\alpha}) + i\,\nabla^2(\tau\zeta T) + 2i\mu\,\dot{b}^{\beta}_{\alpha}(\zeta\overline{T}^{\alpha})_{,\beta} + 2\beta\mu\tau\zeta(\frac{1}{\tau^2\zeta}\overline{T}^{\alpha})_{,\alpha}$$
$$- i\beta\tau\zeta\dot{b}^{\alpha\beta}[\frac{1}{\tau^2\zeta}(\tau\zeta\overline{T}^{\gamma}{}_{,\gamma})_{,\alpha}]_{,\beta} + \beta\tau\zeta\dot{b}^{\alpha\beta}[\frac{1}{\tau^2\zeta}(\tau\zeta T)_{,\alpha}]_{,\beta} =$$
$$\nabla^2(\tau\zeta\nabla^2 P) + i\mu\,\dot{b}^{\alpha\beta}(\zeta P_{,\alpha})_{,\beta} - i\mu\nabla^2(\tau\zeta\rho P) + 2\beta\mu(\zeta\dot{b}^{\gamma}_{\alpha}\, b^{\alpha\beta}P_{,\beta}]_{,\gamma} \tag{41}$$
$$+ \beta\mu\tau\zeta g^{\alpha\beta}(\frac{1}{\tau^2\zeta}P_{,\alpha})_{,\beta} - \beta\mu\tau\zeta\dot{b}^{\alpha\beta}[\frac{1}{\tau^2\zeta}(\tau\zeta\rho P)_{,\alpha}]_{,\beta}$$
$$- i\beta\,\nabla^2[\tau\zeta(b^{\alpha\beta}P_{,\alpha})_{,\beta}] - i\beta\tau\zeta\dot{b}^{\alpha\beta}[\frac{1}{\tau^2\zeta}(\tau\zeta\,\nabla^2 P)_{,\alpha}]_{,\beta} \equiv L(P) .$$

The desired characteristic equation is

$$L(P) = 0 \tag{42}$$

valid for cylinders and shells of revolution. A tentative simplification of this equation, known to be valid in the case of cylinders from the work of Novozhilov, Goldenveizer, and Simmonds is obtained by dropping the β terms. From now on suppose this has been done; the terms can be restored if further investigation proves it necessary.

The details will not be shown here, but if (38) is multiplied by an $O(\beta)$ scalar and added to (39) this modified equation (39) leads, by the above process, to a characteristic equation which differs from (42) only in the $\beta\mu$ terms. Such variability in the $\beta\mu$ terms (not to be mistaken for arbitrariness) was encountered by Simmonds [14] whose paper is enlightening in this regard.

In the case of a cylinder, if one adds $\beta\rho(38)$ to (39), the characteristic equation becomes

$$\nabla^2\left(\frac{1}{\rho}\ \nabla^2 P\right) + 2\beta\mu\left(b^{\alpha\beta}\ P_{,\alpha}\right)_{,\beta} - \frac{i\mu}{\rho}\ \bar{b}^{\alpha\beta}\ P_{,\alpha\beta} = 0 \quad . \tag{43}$$

This equation is the invariant form of Novozhilov's cylinder equation. It can be obtained by elimination from the equations (20) with the last term in each of these equations missing. The result seems to be peculiar to the cylinder. In general, the effect of the last terms in (20) on the characteristic equation (42) cannot be annulled by any such modification of (39).

Once a solution of the characteristic equation has been found λ^1 and λ^2 are given directly in terms of P by (35) and (40) (with $\bar{T}^\alpha = T = 0$). In order to construct $P^{\alpha\beta}$ using (34) one must find λ^3. This can be done using (36), to the first order by quadratures, and more accurately by an iteration process. Complex displacements X_α and W are then found(if necessary) by integration of the equations (15) and (18); which must be possible since the integrability conditions (20) are satisfied. In practice the various integrations mentioned here might prove to be difficult. An alternative formulation of Problem I, possibly more convenient for application, is developed in the next section.

Alternative Characteristic Equation

The expressions (15) for $P^{\alpha\beta}$ and $Q^{\alpha\beta}$ in terms of X_α and W furnish a complete solution to the "equilibrium" equations (13). If one could find expressions for X_α and W, say in terms of some scalar function φ, such that the relation $Q^{\alpha\beta} = - i \overline{P}^{\alpha\beta}$ is satisfied, then one would have a general solution to the equations (20). This is possible as proved in the following.

Let A be a simply connected region of the shell middle surface with boundary C. Let $P^{\alpha\beta}$ be an arbitrary symmetric tensor which is continuous in A together with its first four covariant derivatives and such that $P^{\alpha\beta}$ and its first three covariant derivatives vanish on C. Let T^α and T be defined in terms of $P^{\alpha\beta}$ by (30) and (31). By application of Green's theorem

$$\int_A (T^\alpha X'_\alpha + T W')da = - \int_A (\overline{Q}'_{\alpha\beta} + i P'_{\alpha\beta})P^{\alpha\beta}da \tag{44}$$

in which $\overline{Q}'_{\alpha\beta}$ and $\overline{P}'_{\alpha\beta}$ are related to X'_α and W' as in equations (15). Let $F(T^\alpha,T)$ be the invariant functional of T^α and T on the left-hand side of (41), and let X'_α and W' be defined in A in terms of a function φ by the equations

$$\int_A F(T^\alpha,T)\varphi\, da = - \int_A (T^\alpha X'_\alpha + T W')da . \tag{45}$$

One also has

$$\int_A F\varphi da = \int_A L(P)\varphi da = \int_A \overline{L}(\varphi)\, P\, da \tag{46}$$

where $\tilde{L}$ is the operator adjoint to L. Now if $\mathcal{P}$ satisfies $\tilde{L}(\mathcal{P}) = 0$, then $\int_A (\overline{Q}'_{\alpha\beta} + i\, P'_{\alpha\beta}) P^{\alpha\beta} da = 0$ for arbitrary $P^{\alpha\beta}$ from (44) to (46). It follows that $\overline{Q}'_{\alpha\beta} + i\, P'_{\alpha\beta} = 0$ in A. Explicit expressions for X_α and W obtained by use of (45) and the simplified F are as follows

$$W = i\tau\zeta\nabla^2\mathcal{P} \tag{47}$$

$$X_\alpha = i\, W_{,\alpha} + 2i\mu\zeta(1 - i\beta\rho)\mathring{b}^\beta_\alpha \mathcal{P}_{,\beta} + \frac{4\beta\mu}{\tau\sqrt{\tau\zeta}} (\sqrt{\tau\zeta}\mathcal{P})_{,\alpha} \tag{48}$$

where

$$\tilde{L}(\mathcal{P}) = 0 . \tag{49}$$

These lead (from (15)) to an expression for $\overline{P}_{\alpha\beta}$ in terms of $\mathcal{P}$.

Problem I in the form $Q_{\alpha\beta} = -\, i\, \overline{P}_{\alpha\beta}$, regarded as equations for X_α and W, corresponds to what Novozhilov and Chernykh call "the equations in terms of complex displacements." Such a formulation of the problem has been exploited by Chernykh in his book [3].

A Characteristic Equation for the Spherical Shell

In the case of a sphere $b_{\alpha\beta} = g_{\alpha\beta}$, $B_{\alpha\beta} = 0$ and the preceding analysis breaks down. However, in this particularly simple case it is easy to derive a characteristic equation. For a sphere the equations (19) are

$$(1 + i\beta)P^{\alpha\beta}{}_{,\beta} - i\beta\, g^{\alpha\beta}P_{,\beta} = 0 \tag{50}$$

$$P^{\alpha\beta}{}_{,\alpha\beta} - \nabla^2 P + i\,\mu\, P = 0 \tag{51}$$

from which it follows that

$$\nabla^2 P - i\mu(1 + i\beta)P = 0 . \tag{52}$$

Note that P satisfies a second order equation in this case. The general solution to (50) is

$$P^{\alpha\beta} = \frac{i\beta}{1 + i\beta}\, g^{\alpha\beta}\, P + U^{\alpha\beta} \tag{53}$$

where $U^{\alpha\beta}{}_{,\beta} = 0$. $\quad$ (54)

For a sphere the general solution to (54) is .

$$\bar{U}_{\alpha\beta} = \varphi_{,\alpha\beta} + \beta\mu g_{\alpha\beta}\varphi \tag{55}$$

From (53)

$$P = \frac{1 + i\beta}{1 - i\beta}(\nabla^2\varphi + 2\beta\mu\varphi) \tag{56}$$

and then from (52)

$$[\nabla^2 - i\mu(1 + i\beta)]\ (\nabla^2 + 2\beta\mu)\varphi = 0 \quad . \tag{57}$$

This characteristic equation is identical in form to one derived by Koiter [9] . In terms of φ one has for $\bar{P}_{\alpha\beta}$

$$(1 - i\beta)\bar{P}_{\alpha\beta} = (1 - i\beta)\varphi_{,\alpha\beta} + (1 + i\beta)\beta\mu\ g_{\alpha\beta}\varphi + i\beta\ g_{\alpha\beta}\ \nabla^2\varphi . \tag{58}$$

There are obvious possibilities for simplification. The system of equations for a sphere derived earlier in this paper can be reduced in a similar manner, and the present system can be reduced in other ways.

FURTHER RESULTS ON PROBLEM II

The complete solution to equations (13)* is given by (15)* for $\overset{*}{P}{}^{\alpha\beta}$ and $\overset{*}{Q}{}^{\alpha\beta}$ in terms of $\overset{*}{W}$ and $\overset{*}{X}_\alpha$. Problem II is equivalent to the determination of $\overset{*}{W}$, $\overset{*}{X}_\alpha$, $R^{\alpha\beta}$ and $S^{\alpha\beta}$ such that (16) and (17) are satisfied with order of magnitude restrictions on $R^{\alpha\beta}$ and $S^{\alpha\beta}$ as previously stated. If these restrictions are dropped (for the present) a solution is easily obtained as follows. Write (17) in the form

$$\bar{\overset{*}{Q}}_{\alpha\beta} = - i\ \nu\bar{P}_{\alpha\beta} + i\beta\nu\ \bar{S}_{\alpha\beta} = i\nu[\tfrac{1}{2}(\phi_{\alpha,\beta} + \phi_{\beta,\alpha}) + \beta B_{\alpha\beta}\ z + \beta\ \bar{S}_{\alpha\beta}] \tag{59}$$

and compare to (15)* repeated here

$$\bar{\overset{*}{Q}}_{\alpha\beta} = \tfrac{1}{2}(\overset{*}{X}_{\alpha,\beta} + \overset{*}{X}_{\beta,\alpha}) + \mu\ b_{\alpha\beta}\overset{*}{W} \quad .$$

Equation (59) is satisfied if

$$\overset{*}{X}_\alpha = i\,\nu\phi_\alpha, \quad \overset{*}{W} = 0 \quad \text{and} \quad \bar{S}_{\alpha\beta} = -B_{\alpha\beta}\,z \quad . \tag{60}$$

Now calculate $\overset{*}{\bar{P}}_{\alpha\beta}$ from (15)*, and use (16) to find

$$\bar{R}_{\alpha\beta} = -\frac{1}{2}[(b^\gamma_\alpha\phi_\gamma)_{,\beta} + (b^\gamma_\beta\,\phi_\gamma)_{,\alpha}] - \beta B_{\alpha\beta}(\rho\,z + \frac{1}{2}\,iB^{\gamma\delta}\,\bar{P}_{\gamma\delta}) \quad . \tag{61}$$

This determination of $R_{\alpha\beta}$ and $S_{\alpha\beta}$ furnishes an exact particular solution to the equations (21). The difficulty with it is that the order-of-magnitude requirements are not met in general. By choice of scalin $P_{\alpha\beta}$ and $Q_{\alpha\beta}$ are $0(1)$, and in the usual case $\wp$ is $0(1)$ in equations (47) and (48). From (48)

$$z = \frac{1}{2}\,\varepsilon^{\alpha\beta}X_{\beta,\alpha} \text{ is } 0(\mu) \quad \text{and} \quad S_{\alpha\beta} = B_{\alpha\beta}\,z \text{ is } 0(\mu)$$

rather than $0(1)$ as required. This is unacceptable, in general, because μ can be large. The general solution for $R_{\alpha\beta}$ and $S_{\alpha\beta}$ is obtained by adding the general solution to the homogeneous equations (22) for $R_{\alpha\beta}$ and $S_{\alpha\beta}$ to the particular solution, and since these equations have the same form as (15) a general solution is known. The result is obviously

$$\bar{S}_{\alpha\beta} = -B_{\alpha\beta}z + \frac{1}{2}(X'_{\alpha,\beta} + X'_{\beta,\alpha}) + \mu\,b_{\alpha\beta}W' \tag{62}$$

$$\bar{R}_{\alpha\beta} = -\frac{1}{2}[(\phi'_\alpha + b^\gamma_\alpha\phi_\gamma)_{,\beta} + (\phi'_\beta + b^\gamma_\beta\phi_\gamma)_{,\alpha}] - \beta B_{\alpha\beta}(z' + \rho z + \frac{1}{2}\,i\,B^{\gamma\delta}\bar{P}_{\gamma\delta}) . \tag{63}$$

The complete expressions for $\overset{*}{X}_\alpha$ and $\overset{*}{W}$ used to form $\overset{*}{P}_{\alpha\beta}$ and $\overset{*}{Q}_{\alpha\beta}$ are

$$\overset{*}{X}_\alpha = i\nu(\phi_\alpha + \beta\,X'_\alpha) \tag{64}$$

$$\overset{*}{W} = i\,\beta\,\nu\,W' \tag{65}$$

where X'_α and W' are arbitrary. Problem II will be solved if X'_α and W' can be found to satisfy the order-of-magnitude requirements on $R_{\alpha\beta}$ and $S_{\alpha\beta}$. In the general case the problem in this form does not seem to be an easy one. However, for spheres and cylinders, results for $R_{\alpha\beta}$ and $S_{\alpha\beta}$ without quadratures are readily obtainable.

Spheres

In the case of a sphere $B_{\alpha\beta} = 0$, $b_{\alpha\beta} = g_{\alpha\beta}$ and $X'_\alpha = W' = 0$ leads to the acceptable solution

$$R_{\alpha\beta} = P_{\alpha\beta}\,, \quad S_{\alpha\beta} = 0\ . \tag{66}$$

A more general solution is

$$R_{\alpha\beta} = (1 + C)P_{\alpha\beta}\,, \quad S_{\alpha\beta} = C\,Q_{\alpha\beta} \tag{67}$$

and correspondingly

$$\overset{*}{X}_\alpha = i\nu(\phi_\alpha + \beta\,C\,X_\alpha)\,, \quad \overset{*}{W} = i\,\beta\,\nu\,C\,W \tag{68}$$

where C is any $0(1)$ constant. The arbitrariness involved is simply a manifestation of the well known degree of arbitrariness involved in the form of the shell equations.

Circular Cylinders

In this case put

$$X'_\alpha = b^\gamma_\alpha X_\gamma - X_\alpha\,, \qquad W' = 0\,, \text{ and obtain}$$

$$\begin{aligned} X'_{\alpha,\beta} &= b^\gamma_\alpha X_{\gamma,\beta} - X_{\alpha,\beta} \\ &= b^\gamma_\alpha(\overline{Q}_{\beta\gamma} + \varepsilon_{\beta\gamma}\,z - \mu\,b_{\beta\gamma}\,W) - \overline{Q}_{\alpha\beta} - \varepsilon_{\alpha\beta}\,z + \mu b_{\alpha\beta}\,W \end{aligned}$$

$$\tfrac{1}{2}(X'_{\alpha,\beta} + X'_{\beta,\alpha}) = \tfrac{1}{2}(b^\gamma_\alpha\,\overline{Q}_{\beta\gamma} + b^\gamma_\beta\,\overline{Q}_{\alpha\gamma}) - \overline{Q}_{\alpha\beta} + B_{\alpha\beta}\,z\ .$$

The result for $\overline{S}_{\alpha\beta}$ is

$$\overline{S}_{\alpha\beta} = \tfrac{1}{2}(b^\gamma_\alpha\,\overline{Q}_{\beta\gamma} + b^\gamma_\beta\,\overline{Q}_{\alpha\gamma}) - \overline{Q}_{\alpha\beta}\ . \tag{69}$$

Further

$$\begin{aligned} \phi'_\alpha &= \beta\,b^\gamma_\alpha\,(b^\delta_\gamma\,X_\delta - X_\gamma) = 0 \\ z' &= \tfrac{1}{2}\,\varepsilon^{\alpha\beta} b^\gamma_\beta\,X_{\gamma,\alpha} - z = \tfrac{1}{2}(B^{\alpha\gamma} + \tfrac{1}{2}\varepsilon^{\alpha\gamma})X_{\gamma,\alpha} - z \\ &= \tfrac{1}{2}\,B^{\alpha\gamma}\,\overline{Q}_{\alpha\gamma} - \tfrac{1}{2}\,z\ . \end{aligned}$$

The result for $\overline{R}_{\alpha\beta}$ is

$$\overline{R}_{\alpha\beta} = \tfrac{1}{2}(b^\gamma_\alpha\,\overline{P}_{\gamma\beta} + b^\gamma_\beta\,\overline{P}_{\gamma\alpha}) - \tfrac{3}{2}\,i\beta\,B_{\alpha\beta}\,B^{\gamma\delta}\,\overline{P}_{\gamma\delta} \tag{70}$$

and for $\overset{*}{X}_\alpha$ and $\overset{*}{W}$ there is

$$\overset{*}{X}_\alpha = i\nu(-W_{,\alpha} - \beta X_\alpha + 2\beta\, b^\gamma_\alpha X_\gamma)\ , \quad \overset{*}{W} = 0 \quad . \tag{71}$$

These results can be generalized as in the case of the sphere.

General Cylinders.

For cylinders one has the formulas

$$b^\gamma_{\alpha,\beta} = \frac{1}{\rho}\ b^\gamma_\alpha\, \rho_{,\beta} = \frac{1}{\rho}\, b_{\alpha\beta} g^{\gamma\delta} \rho_{,\delta} = \frac{1}{\rho}\ b_{\alpha\beta}\rho^{,\gamma} \ .$$

The choice

$$\begin{aligned} X'_\alpha &= b^\gamma_\alpha X_\gamma \\ W' &= 2\rho\ W - \frac{1}{\mu\rho}\,\rho_{,\gamma} X^\gamma \end{aligned} \tag{72}$$

leads to the results

$$\begin{aligned} \overset{*}{X}_\alpha &= i\nu(-W_{,\alpha} + 2\beta\, b^\gamma_\alpha X^\gamma) \\ \overset{*}{W} &= i\beta\nu(2\rho\ W - \frac{1}{\mu\rho}\,\rho_{,\gamma} X^\gamma) \\ \overline{S}_{\alpha\beta} &= \frac{1}{2}(b^\gamma_\alpha\, \overline{Q}_{\gamma\beta} + b^\gamma_\beta\, \overline{Q}_{\gamma\alpha}) \end{aligned} \tag{73}$$

$$\begin{aligned} \overline{R}_{\alpha\beta} = {} & 2\rho\ \overline{P}_{\alpha\beta} + \frac{1}{2}(b^\gamma_\alpha\, \overline{P}_{\gamma\beta} + b^\gamma_\beta\, \overline{P}_{\gamma\alpha}) - \frac{3}{2}\, i\beta\ B_{\alpha\beta}\ B^{\gamma\delta}\ \overline{P}_{\gamma\delta} + 2\rho_{,\alpha\beta}\ W \\ & + 2\rho_{,\alpha} W_{,\beta} + 2\rho_{,\beta} W_{,\alpha} + \frac{1}{\rho}\, b_{\alpha\beta}\ \rho^{,\gamma}\ W_{,\gamma} - \frac{1}{\mu}(\frac{1}{\rho}\rho_{,\gamma} X^\gamma)_{,\alpha\beta} - 4\beta\ b_{\alpha\beta}\ \rho_{,\gamma} X^\gamma \ . \end{aligned}$$

The last term in the expression for $\overline{R}_{\alpha\beta}$ is apparently $O(\beta\mu)$ which would violate the order-of-magnitude requirements on $\overline{R}_{\alpha\beta}$ in case $\beta\mu$ is large (assuming such a possibility). A detailed investigation proves that $\beta\rho_{,\gamma} X^\gamma = O(1)$ in any case. The analysis is too lengthy to give here, but a brief statement of the facts may be in order. When $\beta\mu$ is large the characteristic equation is approximately (assuming $\rho_{,y} = O(1)$)

$$2\beta(\rho\,\mathcal{P}_{,y})_{,y} - i\,\mathcal{P}_{,xx} = 0 \tag{74}$$

in cartesian coordinates with x in the axial direction. Evidently there are two length scales involved since $\mathcal{P}_{,x} \ll \mathcal{P}_{,y}$ from (74). To be consistent here the L in μ should be the shorter (circumferential) "wavelength" so that no differentiation increases the order of magnitude of $\wp$. One must use (47) and (48) to calculate X_α, W and from these calculate $\bar{Q}_{\alpha\beta}$. When $\wp$ is scaled to make $\bar{Q}_{\alpha\beta} = O(1)$ it turns out that $\beta\mu\wp = O(1)$ and $\beta X_y = O(1)$. In the unlikely event that $\rho_{,y}$ is large $\wp$ becomes smaller to compensate; in any case $\bar{R}_{\alpha\beta} = O(1)$.

Concluding Remarks

The linear equations of thin shell theory have been reduced in complex form to a true fourth order system (with no conjugation operation) without approximation of the equilibrium or compatibility equation, but at the expense of introducing an auxiliary system of equations for certain allowable error terms in the constitutive relations. The fourth order system has been reduced to a single fourth order equation in certain cases. Other than this no serious attempt has been made at further simplification of the equations (which might be made in general or in specific cases) by dropping small terms. However, such approximations are usually easier to justify in a single equation than in a system of equations. Extensions of the results to include the case of distributed loads remains to be worked out.

A characteristic equation for the membrane-inextensional bending theories (which are combined in the complex formulation) is obtained by omitting all but the μ terms in $L(P) = 0$ or $\overline{L}(\varphi) = 0$. In this connection the author encountered a paradoxical situation. A characteristic equation for the membrane-inextensional bending (M.I.B.) theory in terms of z (the Weingarten equation) can be obtained for any shape of middle surface. A full account of the theory is to be found in Vekua's book [15]. On the other hand, a characteristic equation (for the M.I.B.theory) in terms of P is difficult if not impossible to obtain except in those cases treated in this paper. One might think that a general characteristic equation for the bending theory in terms of z would be easy to obtain. It turns out that the relation $Q_{\alpha\beta} = - i \overline{P}_{\alpha\beta}$ can be replaced by equivalent equations in terms of rotations ϕ_α and z only. In the M.I.B. case $(Q_{\alpha\beta} = 0)\phi_\alpha$ is easily eliminated to obtain the Weingarten equation. The author has been unable to perform the elimination for the bending theory in any case except the circular cylinder, (assuming that higher than fourth derivatives of z are ruled out). An important unanswered question is the following: Can the system (19) be redued to a single fourth order equation in general, and if not, can it be so reduced in cases other than those already found?

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